

A Level Mathematics A
H240/01 Pure Mathematics

Question Set 2

1

In this question you must show detailed reasoning.

Find the two real roots of the equation $x^4 - 5 = 4x^2$. Give the roots in an exact form.

[4]

$$\begin{aligned}
 1. \quad & x^4 - 5 = 4x^2 \\
 & x^4 - 4x^2 - 5 = 0 && A = x^2 \\
 & A^2 - 4A - 5 = 0 \\
 & (A - 5)(A + 1) = 0 \\
 & A - 5 = 0 & A + 1 = 0 \\
 & A = 5 & A = -1 \\
 & x^2 = 5 & x^2 = -1 \\
 & x = \pm\sqrt{5} & \times \text{ as } < 0
 \end{aligned}$$

2 Prove algebraically that $n^3 + 3n - 1$ is odd for all positive integers n .

[4]

$$\begin{aligned}
 2. \quad & n^3 + 3n - 1 \\
 & \text{if } n = \text{even} \rightarrow n \text{ can be written as } 2m \\
 & n^3 + 3n - 1 = 8m^3 + 6m - 1 \\
 & \quad = \underline{2(4m^3 + 3m)} - 1 \\
 & \text{for all } m, \text{ even} \leftarrow \text{hence } 2(4m^3 + 3m) - 1 \text{ is odd} \\
 & \text{if } n = \text{odd} \rightarrow n \text{ can be written as } 2m + 1 \\
 & n^3 + 3n - 1 = (2m + 1)^3 + 3(2m + 1) - 1 \\
 & \quad = (8m^3 + 12m^2 + 6m + 1) + (6m + 3) - 1 \\
 & \quad = 8m^3 + 12m^2 + 12m + 3 \\
 & \quad = \underline{2(4m^3 + 6m^2 + 6m)} + 3 \\
 & \text{for all } m, \text{ even} \leftarrow \text{hence } 2(4m^3 + 6m^2 + 6m) + 3 \text{ is odd}
 \end{aligned}$$

3 The equation of a circle is $x^2 + y^2 + 6x - 2y - 10 = 0$.

(a) Find the centre and radius of the circle.

[3]

$$\begin{aligned} 3. \quad & x^2 + y^2 + 6x - 2y - 10 = 0 \\ \text{a)} \quad & x^2 + 6x + y^2 - 2y - 10 = 0 \\ & (x^2 + 6x) + (y^2 - 2y) - 10 = 0 \\ & (x^2 + 6x + 9 - 9) + (y^2 - 2y + 1 - 1) - 10 = 0 \\ & (x + 3)^2 + (y - 1)^2 - 9 - 1 - 10 = 0 \\ & (x + 3)^2 + (y - 1)^2 = 20 \\ & (x + 3)^2 + (y - 1)^2 = (\sqrt{20})^2 = (2\sqrt{5})^2 \end{aligned}$$

centre $(-3, 1)$ radius $2\sqrt{5}$

(b) Find the coordinates of any points where the line $y = 2x - 3$ meets the circle $x^2 + y^2 + 6x - 2y - 10 = 0$.

[4]

$$\begin{aligned} \text{b)} \quad & x^2 + y^2 + 6x - 2y - 10 = 0 \quad \leftarrow y = 2x - 3 \\ & x^2 + (2x - 3)^2 + 6x - 2(2x - 3) - 10 = 0 \\ & x^2 + 4x^2 - 12x + 9 + 6x - 4x + 6 - 10 = 0 \\ & 5x^2 - 10x + 5 = 0 \\ & x^2 - 2x + 1 = 0 \\ & (x - 1)^2 = 0 \\ & x = 1 \quad \longrightarrow \quad \begin{aligned} & y = 2x - 3 \\ & y = 2(1) - 3 = -1 \end{aligned} \\ & \quad \quad \quad (1, -1) \end{aligned}$$

(c) State what can be deduced from the answer to part (ii) about the line $y = 2x - 3$ and the circle $x^2 + y^2 + 6x - 2y - 10 = 0$.

[1]

c) - line is a tangent to the circle at $(1, -1)$

(a) Find the first three terms in the expansion of $(4-x)^{-\frac{1}{2}}$ in ascending powers of x . [4]

$$\begin{aligned}
 4. \text{ a) } (4-x)^{-\frac{1}{2}} &= 4^{-\frac{1}{2}} \left(1 - \frac{1}{4}x\right)^{-\frac{1}{2}} \\
 &= \frac{1}{2} \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{1}{4}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!} \left(-\frac{1}{4}x\right)^2 + \dots \right] \\
 &= \frac{1}{2} \left[1 + \frac{1}{8}x + \frac{3}{128}x^2 \right] \\
 &= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2
 \end{aligned}$$

(b) The expansion of $\frac{a+bx}{\sqrt{4-x}}$ is $16-x \dots$. Find the values of the constants a and b . [3]

$$b) \quad \frac{a+bx}{\sqrt{4-x}} = 16-x$$

$$(a+bx) (4-x)^{-\frac{1}{2}} = 16-x$$

\uparrow \uparrow
 1st term 2nd term

$$(a+bx) \left(\frac{1}{2} + \frac{1}{16}x\right) = 16-x$$

$$\frac{1}{2}a + \frac{1}{16}ax + \frac{1}{2}bx + \frac{1}{16}bx^2 = 16-x$$

$$\frac{1}{2}a = 16$$

$$a = 32$$

$$\frac{1}{16}a + \frac{1}{2}b = -1$$

$$\frac{1}{16} \times 32 + \frac{1}{2}b = -1$$

$$2 + \frac{1}{2}b = -1$$

$$\frac{1}{2}b = -3$$

$$b = -6$$

5 The function f is defined for all real values of x as $f(x) = c + 8x - x^2$, where c is a constant.

(a) Given that the range of f is $f(x) \leq 19$, find the value of c .

[3]

5. $f(x) = c + 8x - x^2$

a) $-x^2 + 8x + c$
 $= -(x^2 - 8x) + c$
 $= -(x^2 - 8x + 16 - 16) + c$
 $= -(x - 4)^2 + 16 + c$

$f(x) \leq 19$ since $-(x-4)^2$ is negative,
 maximum positive value for $16+c$ is 19

$16 + c = 19$
 $c = 3$

(b) Given instead that $ff(2) = 8$, find the possible values of c .

[4]

b) $f(f(2)) = 8$ $f(x) = c + 8x - x^2$
 $f(f(x)) = c + 8(c + 8x - x^2) - (c + 8x - x^2)^2 = 8$
 $\Rightarrow c + 8(c + 8 \times 2 - 2^2) - (c + 8 \times 2 - 2^2)^2 = 8$
 $= c + 8(c + 12) - (c + 12)^2$
 $= c + 8c + 96 - c^2 - 24c - 144$
 $= -c^2 - 15c - 48 = 8$
 $c^2 + 15c + 56 = 0$
 $(c + 8)(c + 7) = 0$
 $c = -8$ or -7

$c = -7, -8$

6 A curve has parametric equations $x = t + \frac{2}{t}$ and $y = t - \frac{2}{t}$, for $t \neq 0$.

(a) Find $\frac{dy}{dx}$ in terms of t , giving your answer in its simplest form.

[4]

b. $x = t + \frac{2}{t} = t + 2t^{-1}$ $y = t - \frac{2}{t} = t - 2t^{-1}$

a) $\frac{dx}{dt} = 1 - \frac{2}{t^2}$ $\frac{dy}{dt} = 1 + \frac{2}{t^2}$
 $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $\frac{dy}{dx} = \left(1 + \frac{2}{t^2}\right) \left(\frac{1}{1 - \frac{2}{t^2}}\right)$
 $= \left(\frac{t^2 + 2}{t^2}\right) \left(\frac{1}{\frac{t^2 - 2}{t^2}}\right)$

$$= \left(\frac{t^2+2}{t} \right) \left(\frac{t}{t^2-2} \right)$$

$$\frac{dy}{dx} = \frac{t^2+2}{t^2-2}$$

(b) Explain why the curve has no stationary points.

[2]

b) $\frac{dy}{dx} = 0$ at stationary point

$$\frac{t^2+2}{t^2-2} = 0$$

$$t^2+2=0$$

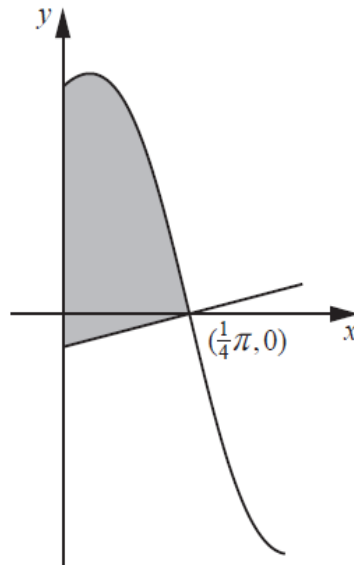
$$t^2 = -2$$

$t^2 \geq 0$ hence $t^2+2=0$ has no solutions

hence curve has no stationary points

7

In this question you must show detailed reasoning.



The diagram shows the curve $y = \frac{4 \cos 2x}{3 - \sin 2x}$, for $x \geq 0$, and the normal to the curve at the point $(\frac{1}{4}\pi, 0)$. Show that the exact area of the shaded region enclosed by the curve, the normal to the curve and the y-axis is $\ln \frac{9}{4} + \frac{1}{128}\pi^2$. [10]

$$\begin{aligned}
 7. \quad y &= \frac{4 \cos 2x}{3 - \sin 2x} & u &= 4 \cos 2x & u' &= -8 \sin 2x \\
 & & v &= 3 - \sin 2x & v' &= -2 \cos 2x \\
 \frac{dy}{dx} &= \frac{(3 - \sin 2x)(-8 \sin 2x) - (4 \cos 2x)(-2 \cos 2x)}{(3 - \sin 2x)^2} \\
 &= \frac{-24 \sin 2x + 8 \sin^2 2x + 8 \cos^2 2x}{(3 - \sin 2x)^2}
 \end{aligned}$$

when $x = \frac{1}{4}\pi$ $\frac{dy}{dx} = -4$ in radians

gradient at normal = $\frac{1}{4}$ as $-4 \times \frac{1}{4} = -1$ $(\frac{1}{4}\pi, 0)$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{1}{4}(x - \frac{1}{4}\pi)$$

$$y = \frac{1}{4}x - \frac{1}{16}\pi$$

when $x = 0$, $y = -\frac{1}{16}\pi$

area of triangle under x-axis

$$\frac{1}{2} \times \frac{1}{4}\pi \times \frac{1}{16}\pi = \frac{1}{128}\pi^2$$

$$\int_0^{\frac{1}{4}\pi} \frac{u \cos 2x}{3 - \sin 2x} dx$$

$$u = 3 - \sin 2x \quad \frac{du}{dx} = -2 \cos 2x$$

$$= \int_3^2 \frac{\frac{2}{u} \times \frac{1}{-2 \cos 2x}}{-2 \cos 2x} du$$

$$dx = \frac{1}{-2 \cos 2x} du$$

$$= \int_3^2 -\frac{2}{u} du = 2 \int_3^2 -\frac{1}{u} du$$

$$u = 3 - \sin 2(\frac{1}{4}\pi) = 2 \quad u = 3 - \sin 2(0) = 3$$

$$= 2[-\ln u]_3^2 = 2[(-\ln 2) - (-\ln 3)] = -2\ln 2 + 2\ln 3$$

$$= \ln 2^{-2} + \ln 3^2 = \ln \frac{1}{4} + \ln 9 = \ln(\frac{1}{4} \times 9) = \ln \frac{9}{4}$$

area between curve & x-axis

$$\ln \frac{9}{4}$$

$$\ln \frac{9}{4} + \frac{1}{128}\pi^2$$

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