A Level Mathematics A

H240/01 Pure Mathematics

Question Set 2

1

In this question you must show detailed reasoning.

Find the two real roots of the equation $x^4 - 5 = 4x^2$. Give the roots in an exact form. **[4]**

1.

$$x^4 - 5 = 4x^2$$

$$x^4 - 4x^2 - 5 = 0 \qquad A = x^2$$

$$A^2 - 4A - 5 = 0$$

$$(A-5)(A+1) = 0$$

$$A - 5 = 0 \qquad A + 1 = 0$$

$$A = 5 \qquad A = -1$$

$$x^2 = 5 \qquad x^2 = -1$$

$$x = \pm\sqrt{5} \qquad \times \text{ as } < 0$$

2 Prove algebraically that $n^3 + 3n - 1$ is odd for all positive integers n. **[4]**

2. $n^3 + 3n - 1$

if n = even $\rightarrow$ n can be written as $2m$

$$n^3 + 3n - 1 = 8m^3 + 6m - 1$$

$$= \underbrace{2(4m^3 + 3m)}_{} - 1$$

for all m, even

hence $2(4m^3 + 3m) - 1$ is odd

if n = odd $\rightarrow$ n can be written as $2m + 1$

$$n^3 + 3n - 1 = (2m+1)^3 + 3(2m+1) - 1$$

$$= (8m^3 + 12m^2 + 6m + \cancel{1}) + (6m + 3) \cancel{-1}$$

$$= 8m^3 + 12m^2 + 12m + 3$$

$$= \underbrace{2(4m^3 + 6m^2 + 6m)}_{} + 3$$

for all m, even

hence $2(4m^3 + 6m^2 + 6m) + 3$ is odd

3 The equation of a circle is $x^2+y^2+6x-2y-10=0$.

(a) Find the centre and radius of the circle. **[3]**

3. $x^2+y^2+6x-2y-10=0$

a) $x^2+6x+y^2-2y-10=0$

$(x^2+6x)+(y^2-2y)-10=0$

$(x^2+6x+9-9)+(y^2-2y+1-1)-10=0$

$(x+3)^2+(y-1)^2-9-1-10=0$

$(x+3)^2+(y-1)^2=20$

$(x+3)^2+(y-1)^2=(\sqrt{20})^2=(2\sqrt{5})^2$

centre $(-3,1)$ radius $2\sqrt{5}$

(b) Find the coordinates of any points where the line $y=2x-3$ meets the circle $x^2+y^2+6x-2y-10=0$. **[4]**

b) $x^2+y^2+6x-2y-10=0 \leftarrow y=2x-3$

$x^2+(2x-3)^2+6x-2(2x-3)-10=0$

$x^2+4x^2-12x+9+6x-4x+6-10=0$

$5x^2-10x+5=0$

$x^2-2x+1=0$

$(x-1)^2=0$

$x=1 \longrightarrow y=2x-3$

$y=2(1)-3=-1$

$(1,-1)$

(c) State what can be deduced from the answer to part **(ii)** about the line $y=2x-3$ and the circle $x^2+y^2+6x-2y-10=0$. **[1]**

c) - line is a tangent to the circle at $(1,-1)$

4

(a) Find the first three terms in the expansion of $(4-x)^{-\frac{1}{2}}$ in ascending powers of x. [4]

4. a) $(4-x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1-\frac{1}{4}x\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{1}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{4}x\right)^2+\ldots\right]$

$= \frac{1}{2}\left[1+\frac{1}{8}x+\frac{3}{128}x^2\right]$

$= \frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2$

(b) The expansion of $\frac{a+bx}{\sqrt{4-x}}$ is $16-x \ldots$. Find the values of the constants a and b. [3]

b) $\frac{a+bx}{\sqrt{4-x}} = 16-x$

$(a+bx)(4-x)^{-\frac{1}{2}} = \underset{\text{1st term}}{16} - \underset{\text{2nd term}}{x}$

$(a+bx)\left(\frac{1}{2}+\frac{1}{16}x\right) = 16-x$

$\frac{1}{2}a + \frac{1}{16}ax + \frac{1}{2}bx + \frac{1}{16}bx^2 = 16-x$

$\frac{1}{2}a = 16$

$\boxed{a = 32}$

$\frac{1}{16}a + \frac{1}{2}b = -1$

$\frac{1}{16}\times 32 + \frac{1}{2}b = -1$

$2 + \frac{1}{2}b = -1$

$\frac{1}{2}b = -3$

$\boxed{b = -6}$

5 The function f is defined for all real values of x as $f(x) = c + 8x - x^2$, where c is a constant.

(a) Given that the range of f is $f(x) \leqslant 19$, find the value of c. **[3]**

5. $f(x) = c + 8x - x^2$

a) $-x^2 + 8x + c$

$= -(x^2 - 8x) + c$

$= -(x^2 - 8x + 16 - 16) + c$

$= -(x-4)^2 + 16 + c$

$f(x) \leq 19$

since $-(x-4)^2$ is negative, maximum positive value for $16+c$ is 19

$16 + c = 19$

$c = 3$

(b) Given instead that $ff(2) = 8$, find the possible values of c. **[4]**

b) $f(f(2)) = 8$

$f(x) = c + 8x - x^2$

$f(f(x)) = c + 8(c + 8x - x^2) - (c + 8x - x^2)^2 = 8$

$\Rightarrow c + 8(c + 8\times 2 - 2^2) - (c + 8\times 2 - 2^2)^2$

$= c + 8(c + 12) - (c + 12)^2$

$= c + 8c + 96 - c^2 - 24c - 144$

$= -c^2 - 15c - 48 = 8$

$c^2 + 15c + 56 = 0$

$(c+8)(c+7) = 0$

$c = -8$ or -7

$c = -7, -8$

6 A curve has parametric equations $x = t + \frac{2}{t}$ and $y = t - \frac{2}{t}$, for $t \neq 0$.

(a) Find $\frac{dy}{dx}$ in terms of t, giving your answer in its simplest form. **[4]**

6. $x = t + \frac{2}{t} = t + 2t^{-1}$

$y = t - \frac{2}{t} = t - 2t^{-1}$

a) $\frac{dx}{dt} = 1 - \frac{2}{t^2}$

$\frac{dy}{dt} = 1 + \frac{2}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \left(1 + \frac{2}{t^2}\right)\left(\frac{1}{1 - \frac{2}{t^2}}\right)$

$= \left(\frac{t^2+2}{t^2}\right)\left(\frac{1}{\frac{t^2-2}{t^2}}\right)$

$$= \left(\frac{t^2+2}{\cancel{t}}\right)\left(\frac{\cancel{t}}{t^2-2}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{t^2+2}{t^2-2}}$$

(b) Explain why the curve has no stationary points. **[2]**

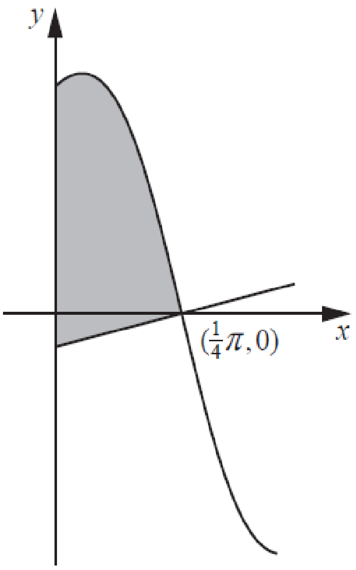
b) $\frac{dy}{dx} = 0$ at stationary point

$\frac{t^2+2}{t^2-2} = 0$ $\quad t^2+2=0 \quad t^2=-2$

$t^2 \geq 0$ hence $t^2+2=0$ has no solutions

hence curve has no stationary points

7 **In this question you must show detailed reasoning.**



The diagram shows the curve $y = \frac{4\cos 2x}{3-\sin 2x}$, for $x \geqslant 0$, and the normal to the curve at the point $(\frac{1}{4}\pi, 0)$. Show that the exact area of the shaded region enclosed by the curve, the normal to the curve and the y-axis is $\ln\frac{9}{4} + \frac{1}{128}\pi^2$. **[10]**

7. $y = \frac{4\cos 2x}{3-\sin 2x}$ $\quad u = 4\cos 2x \quad u' = -8\sin 2x$

$\quad v = 3 - \sin 2x \quad v' = -2\cos 2x$

$$\frac{dy}{dx} = \frac{(3-\sin 2x)(-8\sin 2x) - (4\cos 2x)(-2\cos 2x)}{(3-\sin 2x)^2}$$

$$= \frac{-24\sin 2x + 8\sin^2 2x + 8\cos^2 2x}{(3-\sin 2x)^2}$$

when $x = \frac{1}{4}\pi$ $\qquad \frac{dy}{dx} = -4$ $\qquad$ in radians

gradient at normal $= \frac{1}{4}$ $\qquad$ as $-4 \times \frac{1}{4} = -1$ $\qquad (\frac{1}{4}\pi, 0)$

$y - y_1 = m(x - x_1)$ $\qquad y - 0 = \frac{1}{4}(x - \frac{1}{4}\pi)$

$y = \frac{1}{4}x - \frac{1}{16}\pi$

when $x = 0$, $\quad y = -\frac{1}{16}\pi$

area of triangle under x-axis

$\frac{1}{2} \times \frac{1}{4}\pi \times \frac{1}{16}\pi = \frac{1}{128}\pi^2$

$\int_0^{\frac{1}{4}\pi} \frac{4\cos 2x}{3 - \sin 2x}\,dx$ $\qquad u = 3 - \sin 2x$ $\qquad \frac{du}{dx} = -2\cos 2x$

$= \int_3^2 \frac{\cancel{4\cos 2x}^{\,2}}{u} \times \frac{1}{-\cancel{2\cos 2x}}\,du$ $\qquad dx = \frac{1}{-2\cos 2x}\,du$

$= \int_3^2 -\frac{2}{u}\,du = 2\int_3^2 -\frac{1}{u}\,du$ $\qquad u = 3 - \sin 2(\frac{1}{4}\pi) = 2$ $\qquad u = 3 - \sin 2(0) = 3$

$= 2[-\ln u]_3^2 = 2[(-\ln 2) - (-\ln 3)] = -2\ln 2 + 2\ln 3$

$= \ln 2^{-2} + \ln 3^2 = \ln \frac{1}{4} + \ln 9 = \ln(\frac{1}{4} \times 9) = \ln \frac{9}{4}$

area between curve & x-axis

$\ln \frac{9}{4}$ $\qquad \ln\frac{9}{4} + \frac{1}{128}\pi^2$

Total Marks for Question Set 2: 50 Marks